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$SU_{q,\hbar\to 0}(2)$ and $SU_{q,\hbar}(2)$, the classical and quantum q-deformations of the SU(2) algebra: II. The Hopf algebra, the Yang–Baxter equation and multi-deformed algebraic structures

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Abstract. the $SU_q(2)$ algebra is realized by means of both the Poisson brackets in classical mechanics and commutators in quantum mechanics in a system with q-deformed oscillators of two different types. The structures of the Hopf algebra and the quantum Yang-Baxter equation are also discussed on a quantum level. A set of j-representations of the quantum algebra $SU_q(2)$ is constructed based on the 'type-II' q-oscillators. Multi-deformations of the oscillators of the two types and multi-deformed algebras expressed in Poisson as well as in Lie brackets are proposed.

1. Introduction

Conventionally, the quantum group [1–4] is the Hopf algebra which is neither commutative nor co-commutative. Thus it can be treated as the functional ring in non-commutative, i.e. quantum, space. From this point of view, a quantum group is the symmetry group in quantum space. Here the word 'quantum' can have two meanings, one is from canonical quantization in quantum mechanics, which is supplemented by the Planck constant $\hbar$; another is from the Yang–Baxter equation (YBE) [5] rooted in many problems in quantum statistics. Also, solutions of classical YBE are closely related with classical or semi-simple groups, but solutions of quantum YBE related with quantum groups. These two meanings are keenly interrelated.

However, as we have stressed in earlier work [6, 7]†, the $SU_q(2)$ algebra can be realized in a classical mechanical system, and this algebra is expressed in Poisson brackets and is denoted as $SU_{q,\hbar\to 0}(2)$ in [7] (hereafter referred to as (I)). Through canonical quantization, it becomes a quantum mechanical system, and the realization of $SU_q(2)$ by Poisson bracket, is canonically quantized to form conventional $SU_q(2)$ algebra expressed in Lie brackets, and is denoted as $SU_{q,\hbar}(2)$ in (I). As we emphasized in (I), the q-deformation and the canonical quantization in quantum groups are different from and independent of each other in principle, although q-deformations of quantum mechanical systems are widely discussed.

† Reference [7] is the first of a series of papers on the classical and quantum q-deformations of the SU(2) algebra.

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In this paper, we put forward another model of q-deformed oscillators, based an on undeformed phase space V with undeformed symplectic form Ω, i.e. on the phase space and the symplectic structure of undeformed linear harmonic oscillators. In order to distinguish the second model from the first one which is discussed in [6–11], we call the first 'type I', and the second 'type II'. In this paper, the $\mathrm{SU}_{q,\hbar\to 0}(2)$ and $\mathrm{SU}_{q,\hbar}(2)$ algebras are realized with the type-II q-deformed oscillators in classical and quantum systems respectively, and the latter can be reached through canonical quantization of the system as well. We also show that similar to the case of type I in (I) the canonical motion of the type-II q-deformed oscillators in classical mechanics is exactly solvable, which is oscillation with frequency depending on amplitude.

One of the tasks in this paper is to set up the Hopf algebraic structure and Yang–Baxter relation based upon the quantum q-deformed oscillators of both type I and type II. To this end we start by defining the co-multiplication, co-unit and antipodal map, and then give the Hopf algebras with multiplication operation. The derivation of the corresponding YBE is also shown.

Another major purpose of this paper is to present a kind of new algebraic structures: the multi-deformations of the SU(2) algebra in Poisson brackets as well as in Lie brackets. As one is aware that the single-deformations of Poisson and Lie algebras are quantum algebras, it is natural to study such a problem. Actually, the multi-deformations of the ordinary algebra SU(2), in Poisson and Lie brackets, denoted as $\mathrm{SU}_{q_1 q_2 \cdots q_n,\hbar\to 0}(2)$ and $\mathrm{SU}_{q_1 q_2 \cdots q_n,\hbar}(2)$, respectively, when adding an extra generator H, i.e. the Hamiltonian of the undeformed system, do form a kind of new algebraic structure. In particular cases, the multi-deformed algebras go back to the ordinary algebras SU(2), in Poisson and Lie brackets. It is of interest that the multi-deformed algebras based upon the two different types of q-deformed oscillators are not the same, although the single-deformed algebras for the both cases are the same.

This paper is arranged as follows. In section 2, we give a realization of $\mathrm{SU}_{q,\hbar\to 0}(2)$ in a classical mechanical system with q-deformed oscillators of the second type, in symplectic space (V,Ω) without deformation. The equations of canonical motion and their solutions are given. Section 3 is devoted to the canonical quantization to the system of q-oscillators, and the realization of $\mathrm{SU}_{q,\hbar}(2)$, as well as the Hopf algebra and quantum YBE by means of the quantum q-oscillators. In a certain metric, we show a set of orthogonal basis for $\mathrm{SU}_q(2)$ based upon the oscillators of type II. In section 4, The algebraic structures realized through multi-deformations are given for the cases of type I and two oscillators respectively. Finally, in section 5, we conclude with a brief discussion and make some remarks.

2. $\mathrm{SU}_{q,\hbar\to 0}(2)$ via classical q-oscillators

As in (I), we stress that the q-deformation of a set of classical oscillators leads to $\mathrm{SU}_{q,\hbar\to 0}(2)$ symmetry via Poisson brackets, and through canonical quantization, we arrive at a quantum mechanical system with $\mathrm{SU}_{q,\hbar}(2)$ symmetry in Lie brackets. Now, we present the type-II q-deformed oscillators, and analyse how to realize the q-deformed algebra $\mathrm{SU}_{q,\hbar\to 0}(2)$ and $\mathrm{SU}_{q,\hbar}(2)$ with type-II oscillators.

Let us start with a classical mechanical system of oscillators with the following Hamiltonian

$$H = \sum_{i=1,2} \frac{1}{2}(p_i^2 + q_i^2) \tag{2.1}$$

where the mass m and frequency ω_0 are taken to be 1. The symplectic structure in the phase space V is given by

$$\Omega = \sum_{i=1,2} \mathrm{d}q_i \wedge \mathrm{d}p_i. \tag{2.2}$$

We introduce complex quantities z_i and $\bar{z}_i$

$$z_i = \frac{p_i + \mathrm{i}q_i}{\sqrt{2}} \qquad \bar{z}_i = \frac{p_i - \mathrm{i}q_i}{\sqrt{2}} \tag{2.3}$$

so that (2.1) and (2.2) are rewritten as

$$H = \sum_{i=1,2} z_i \bar{z}_i \tag{2.4}$$

$$\Omega = -\mathrm{i} \sum_{i=1,2} \mathrm{d}z_i \wedge \mathrm{d}\bar{z}_i. \tag{2.5}$$

A set of observables can be constructed on the phase space (V, Ω)

$$J_+ = z_1 \bar{z}_2 \qquad J_- = z_2 \bar{z}_1 \qquad J_3 = \tfrac{1}{2}(z_1\bar{z}_1 - z_2\bar{z}_2). \tag{2.6}$$

It can be easily checked that the observables $J_\pm$ and J_3 satisfy the following relations in Poisson brackets:

$$\{J_+, J_-\} = (-\mathrm{i})2J_3 \qquad \{J_3, J_\pm\} = (-\mathrm{i})(\pm J_\pm) \tag{2.7}$$

which is just the SU(2) algebra expressed by Poisson brackets. Here the following basic Poisson brackets, defined by the symplectic form Ω have been applied:

$$\{z_i, \bar{z}_i\} = -\mathrm{i}\delta_{ij} \qquad \{z_i, z_j\} = 0 \qquad \{\bar{z}_i, \bar{z}_j\} = 0. \tag{2.8}$$

Now we introduce the type-II q-deformed oscillators described by new variables on (V, Ω)

$$z_i' = \frac{1}{\sqrt{\gamma \sinh \gamma}} \frac{\sinh(\gamma z_i \bar{z}_i)}{z_i \bar{z}_i} z_i \qquad \bar{z}_i' = \frac{1}{\sqrt{\gamma \sinh \gamma}} \frac{\sinh(\gamma z_i \bar{z}_i)}{z_i \bar{z}_i} \bar{z}_i \tag{2.9}$$

where $\gamma = \log q$ and $i = 1, 2$ without summation. Also, a set of new observables can be constructed based upon deformed and undeformed oscillators

$$J_+' = z_1 \bar{z}_2' \qquad J_-' = z_2 \bar{z}_1' \qquad J_3' = J_3 = \tfrac{1}{2}(z_1\bar{z}_1 - z_2\bar{z}_2). \tag{2.10}$$

Note that only the observables $J_\pm'$ are differ under deformation from those of the q-deformed oscillators of type I given in (I), but that the phase space (V, Ω) is free of deformation, as in the case of type-I q-oscillators in (I). It can be straightforwardly shown that these observables obey the following relations:

$$\{J_+', J_-'\} = -\mathrm{i}[2J_3'] \qquad \{J_3', J_\pm'\} = -\mathrm{i}(\pm J_\pm') \tag{2.11}$$

where $[x] = \sinh(\gamma x)/\sinh\gamma$. This is just the classical q-deformed SU(2) algebra found in [6] and (I). But this time the algebra is realized by the q-oscillators of type II rather than type I in (I).

To end this section, let us study the dynamics of the q-deformed oscillators of type II, in parallel to that of the type-I q-deformed oscillators provided in (I). The Hamiltonian of the type-II q-deformed oscillators is

$$H' = z_1'\bar{z}_1' + z_2'\bar{z}_2' = \frac{1}{\gamma\sinh\gamma}\left(\frac{\sinh^2\gamma z_1\bar{z}_1}{z_1\bar{z}_1} + \frac{\sinh^2\gamma z_2\bar{z}_2}{z_2\bar{z}_2}\right). \qquad (2.12)$$

Also we can show easily that

$$\{H', J'^2\} = 0 \qquad \{H', J_3'\} = 0 \qquad (2.13)$$

which have counterparts in the quantum theory, i.e. the quantum operators H', J'^2 and J_3' commute and hence have simultaneous eigenstates.

The Hamiltonian in (2.12) gives rise to canonical equations of motion

$$\frac{\mathrm{d}p_i}{\mathrm{d}t} = \{p_i, H'\} = -\frac{1}{\gamma\sinh\gamma}\frac{\gamma B\sinh(2\gamma B) - \sinh^2(\gamma B)}{B^2}q_i \qquad (2.14)$$

$$\frac{\mathrm{d}q_i}{\mathrm{d}t} = \{q_i, H'\} = \frac{1}{\gamma\sinh\gamma}\frac{\gamma B\sinh(2\gamma B) - \sinh^2(\gamma B)}{B^2}p_i. \qquad (2.15)$$

where $B = \frac{1}{2}(p_i^2 + q_i^2)$.

exact solutions exist for these two equations of motion as follows:

$$\begin{aligned} p_i &= A_i\cos\left\{\left[\frac{2}{\gamma\sinh\gamma}\frac{\gamma A_i^2\sinh(\gamma A_i^2) - 2\sinh^2(\gamma\frac{1}{2}A_i^2)}{A_i^4}\right]t\right\} \\ q_i &= A_i\sin\left\{\left[\frac{2}{\gamma\sinh\gamma}\frac{\gamma A_i^2\sinh(\gamma A_i^2) - 2\sinh^2(\gamma\frac{1}{2}A_i^2)}{A_i^4}\right]t\right\}. \end{aligned} \qquad (2.16)$$

Obviously, the motion of the classical type-II q-deformed oscillators is still harmonic with the frequencies depending on the amplitudes, similar to the result for the q-deformed oscillators of type I shown in [6] and (I). But the relation between the frequencies and the amplitudes is more complicated than the former one

$$\omega_i' = \frac{2}{\gamma\sinh\gamma}\frac{\gamma A_i^2\sinh(\gamma A_i^2) - 2\sinh^2(\gamma\frac{1}{2}A_i^2)}{A_i^4}\omega_0 \qquad (2.17)$$

and as we noted at the beginning, ω_0 is taken to be 1.

3. $\mathrm{SU}_{q,\hbar}(2)$, the Hopf algebra and the quantum Yang–Baxter equation *à la* q-oscillators

In the last section, we discussed the realization of the classical q-deformed algebra $\mathrm{SU}_{q,\hbar\to 0}(2)$, the Hopf algebra and 'quantum' Yang–Baxter equation in a system of undeformed and q-deformed and undeformed oscillators of two different types. In this section, we study the canonical quantization of the system of type-II q-deformed oscillators and prove that the quantized system leads to $\mathrm{SU}_{q,\hbar}(2)$. We also show the Hopf algebra and quantum Yang–Baxter equation for $\mathrm{SU}_q(2)$ via both type-I and type-II q-deformed oscillators.

As we remarked, the undeformed observables $J_\pm$ and J_3, and the deformed ones $J'_\pm$, J'_3 are all defined on the phase space (V,Ω). Therefore the canonical quantization should be carried directly out by replacing the basic Poisson brackets by basic commutators for operators, while the variables $\{z_i, \bar{z}_i, z'_i, \bar{z}'_i\}$ are replaced by operators $\{a_i^\dagger, a_i, a_i'^\dagger, a'_i\}$, respectively. The basic commutators are

$$[a_i, a_j^\dagger] = \delta_{ij} \qquad [a_i, a_j] = [a_i^\dagger, a_j^\dagger] = 0. \tag{3.1}$$

where $i, j = 1, 2$. For the undeformed observables $J_\pm$ and J_3, their quantum counterparts

$$J_+ = a_1^\dagger a_2 \qquad J_- = a_2^\dagger a_1 \qquad J_3 = \tfrac{1}{2}(a_1^\dagger a_1 - a_2^\dagger a_2). \tag{3.2}$$

which form the SU(2) algebra

$$[J_3, J_\pm] = \pm J_\pm \qquad [J_+, J_-] = 2J_3. \tag{3.3}$$

As is well known for the harmonic oscillator representation of SU(2), we have a complete set of eigenfunctions

$$|j,m\rangle = \frac{(a_1^\dagger)^{j+m}(a_2^\dagger)^{j-m}}{\sqrt{(j+m)!(j-m)!}}|0\rangle \tag{3.4}$$

which satisfy

$$J_\pm|j,m\rangle = \sqrt{(j\mp m)(j\pm m+1)}\,|j,m\pm 1\rangle \qquad J_3|j,m\rangle = m|j,m\rangle. \tag{3.5}$$

The inner product (or metric) is defined in the following way:

$$\langle f|g\rangle = \langle 0|f(a_1, a_2)g(a_1^\dagger, a_2^\dagger)|0\rangle. \tag{3.6}$$

So the states (3.4) are orthogonal and can be normalized. That is they are a set of orthogonal (and unitary, if one so requires) basis of the Lie algebra SU(2).

While quantizing the deformed oscillators in this system, we have to choose a proper normal ordering. It may be so chosen that

$$\begin{aligned} a_i'^\dagger &= \frac{\sinh(\gamma N_i)}{\sqrt{\gamma \sinh\gamma}\, N_i} a_i^\dagger = a_i^\dagger \frac{\sinh(\gamma(N_i+1))}{\sqrt{\gamma\sinh\gamma}(N_i+1)} \\ a'_i &= a_i \frac{\sinh(\gamma N_i)}{\sqrt{\gamma\sinh\gamma}\, N_i} = \frac{\sinh(\gamma(N_i+1))}{\sqrt{\gamma\sinh\gamma}(N_i+1)} a_i \end{aligned} \tag{3.7}$$

where $N_i = a_i^\dagger a_i$ is particle number operator for the ith ordinary harmonic oscillator. From the above relations, one can see without any difficulty that

$$\begin{aligned} a'^\dagger_i a'_i &= \frac{\sinh\gamma}{\gamma}\frac{[N_i]^2}{N_i} \\ a'_i a'^\dagger_i &= \frac{\sinh\gamma}{\gamma}\frac{[N_i+1]^2}{N_i+1} \\ [a'_i, a'^\dagger_j] &= \frac{\sinh\gamma}{\gamma}\left(\frac{[N_i+1]^2}{N_i+1} - \frac{[N_i]^2}{N_i}\right)\delta_{ij}. \end{aligned} \tag{3.8}$$

Needless to say, this is clearly a new model of q-oscillators which is different from the case given in (I).

The quantum counterparts of the deformed observables (2.10) are

$$J'_+ = a_1^\dagger a'_2 \qquad J'_- = a_2^\dagger a'_1 \qquad J'_3 = \tfrac{1}{2}(a_1^\dagger a_1 - a_2^\dagger a_2) \tag{3.9}$$

and form the quantum algebra

$$[J'_+, J'_-] = \frac{\sinh\gamma}{\gamma}[2J'_3] \qquad [J'_3, J'_\pm] = \pm J'_\pm . \tag{3.10}$$

As in (I), we call this algebra $\mathrm{SU}_{q,\hbar}(2)$, differing from $\mathrm{SU}_{q,\hbar\to 0}(2)$, which is expressed in Poisson brackets. The reader may have noticed that the factor $\sinh\gamma/\gamma$ does not appear in some of the literature. But, it has no essential meaning. Actually, this factor disappears when one rescales the operators J'_+ and J'_-.

Let us now construct the representation space with the Fock space of the system with type-II q-oscillator, i.e.

$$|j,m\rangle = \frac{(a_1^\dagger)^{(j+m)}(a_2^\dagger)^{(j-m)}}{\sqrt{[j+m]!\,[j-m]!}}|0\rangle \tag{3.11}$$

and then we have

$$\begin{aligned} J'_\pm|j,m\rangle &= \sqrt{[j \mp m][j \pm m + 1]}\,|j, m \pm 1\rangle \\ J'_3|j,m\rangle &= m|j,m\rangle \end{aligned} \tag{3.12}$$

which are the q-deformed counterparts of (3.4) and (3.5).

The inner product may be given by

$$\langle f|g\rangle = \langle 0|f(a'_1, a'_2)g(a_1^\dagger, a_2^\dagger)|0\rangle . \tag{3.13}$$

the states given in (3.11) are a set of orthogonal states and can be normalized, i.e. they form a q-deformed j-representation.

Since the q-deformed SU(2) algebra is the same for the oscillators of types I and II, the following constructions of the Hopf algebra and the Yang–Baxter equation are applicable to both cases. Let us first set up the Hopf algebraic structures of $\mathrm{SU}_{q,\hbar}(2)$. Denote the set of operators $J'_\pm, J'_3$ and 1 by A. The brackets of operators are compatible with the definition of the multiplication†

$$m : A \otimes A \to A . \tag{3.14}$$

Besides the multiplication, one may define the co-multiplication and the antipodal map, as well as the co-unit. The co-multiplication $\Delta : A \to A \otimes A$ is defined by

$$\begin{aligned} \Delta(J'_3) &= J'_3 \otimes 1 + 1 \otimes J'_3 \\ \Delta(J'_\pm) &= J'_\pm \otimes q^{J'_3} + q^{-J'_3} \otimes J'_\pm \end{aligned} \tag{3.15}$$

† Hereafter in this section we use the same notation as that used in [17].

the antipode $\gamma : A \to A$ acts as

$$\gamma(J'_3) = -J'_3 \qquad \gamma(J'_\pm) = -q^{\pm 1} J'_\pm \tag{3.16}$$

and the co-unit ϵ: $A \to C$ reads

$$\epsilon(J'_\pm) = \epsilon(J'_3) = 0 \qquad \epsilon(1) = 1. \tag{3.17}$$

It is not difficult to check that the above defined co-multiplication Δ and co-unit ϵ are algebraically the homomorphism $\Delta(ab) = \Delta(a)\Delta(b)$, $\epsilon(ab) = \epsilon(a)\epsilon(b)$. The antipodal mapping γ is algebraically the anti-homomorphism $\gamma(ab) = \gamma(b)\gamma(a)$. The four operations also satisfy the following relations:

$$\begin{aligned}
&(\mathrm{id} \otimes \Delta)\Delta(a) = (\Delta \otimes \mathrm{id})\Delta(a) \\
&m(\mathrm{id} \otimes \gamma)\Delta(a) = m(\gamma \otimes \mathrm{id})\Delta(a) = \epsilon(a) \cdot 1 \\
&(\epsilon \otimes \mathrm{id})\Delta(a) = (\mathrm{id} \otimes \epsilon)\Delta(a) = a
\end{aligned} \tag{3.18}$$

where $a, b, c \in A$. In other words, $(A, m, \Delta, \gamma, \epsilon)$ satisfies all the axioms of a Hopf algebra, but is set up by q-oscillators.

Now we are ready to construct the R matrices satisfying the quantum Yang–Baxter equation based upon the Hopf algebraic structure in the Lie brackets. We first define permutation mapping σ

$$\sigma : A \otimes A \to A \otimes A \qquad \sigma(a \otimes b) = b \otimes a \tag{3.19}$$

and introduce

$$R = q^{J'_3 \otimes J'_3} \sum_{n \geq 0} \frac{(1 - q^{-2})^n}{[n]!} q^{-n(n-1)/2} q^{nJ'_3} (J'_+)^n \otimes q^{-nJ'_3} (J'_-)^n \tag{3.20}$$

then we have

$$\sigma \cdot \Delta(a) = R\Delta(a)R^{-1} \tag{3.21}$$

and

$$\begin{aligned}
&(\mathrm{id} \otimes \Delta)(R) = R_{13}R_{12} \\
&(\Delta \otimes \mathrm{id})(R) = R_{13}R_{23} \\
&(\gamma \otimes \mathrm{id})(R) = R^{-1}.
\end{aligned} \tag{3.22}$$

In other words, $(A; m, \Delta, \epsilon, \gamma, \sigma; R)$ is a quasi-triangular Yang–Baxter algebra. From (3.21) and (3.22), we easily obtain

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12} \tag{3.23}$$

which is just the quantum Yang–Baxter equation.

4. Multi-deformed SU(2) algebraic structures

As was briefly introduced in section 1, the q-deformation can be performed repeatedly on the already q-deformed oscillators. Multi-deformations are obviously meaningful and generically, they can be with different q-parameters. Thus we have a chain for the relations between the variables in the different deformations:

$$(z_i, \bar{z}_i) \xrightarrow{q_1} (z_i', \bar{z}_i') \xrightarrow{q_2} (z_i'', \bar{z}_i'') \xrightarrow{q_3} \cdots \xrightarrow{q_n} (z_i^{(n)}, \bar{z}_i^{(n)}) \xrightarrow{q_{n+1}} \cdots \tag{4.1}$$

As in the single-deformation case, neither the phase space nor the symplectic structure is changed. The only things that undergo deformation are the generators $J_\pm$. In the following, we assume that q is real for simplicity. It can be easily verified that the algebra generated by $J_\pm^{(n)}$, $J_3^{(n)}$ and H stands for both types of q-deformed oscillators at the classical as well as the quantum level. We stress, however, that the multi-deformed algebras realized by type-I and type-II q-oscillators are different in appearance, although in nature they have common points.

Firstly, recall that for the classical q-deformed oscillators discussed in (I), the complex variables are†

$$z_i' = \sqrt{\frac{[N_i]_{\gamma_1}}{N_i}}\, z_i \qquad \bar{z}_i' = \sqrt{\frac{[N_i]_{\gamma_1}}{N_i}}\, \bar{z}_i \tag{4.3}$$

where $\gamma_1 = \log q_1$, $N_i = z_i \bar{z}_i$ and new denotation is used for simplicity, $[x]_{\gamma_1} = \sqrt{(\gamma_1/\sinh\gamma_1)}[x]$. Then we introduce multi-deformed variables for this system

$$z_i^{(n)} = \sqrt{\frac{[N_i^{(n-1)}]_{\gamma_n}}{N_i}}\, z_i^{(n-1)} \qquad \bar{z}_i^{(n)} = \sqrt{\frac{[N_i^{(n-1)}]_{\gamma_n}}{N_i}}\, \bar{z}_i^{(n-1)}. \tag{4.4}$$

Observables can be made in this n-fold deformed system

$$J_+^{(n)} = z_1^{(n)} \bar{z}_2^{(n)} \qquad J_-^{(n)} = z_2^{(n)} \bar{z}_1^{(n)} \qquad J_3^{(n)} = \tfrac{1}{2}(z_1\bar{z}_1 - z_2\bar{z}_2). \tag{4.5}$$

When supplemented by the Hamiltonian H for the undeformed system, the algebra of the observables defined above can be found by straightforward calculation

$$\{H, J_\pm^{(n)}\} = 0 \qquad \{H, J_3^{(n)}\} = 0 \qquad \{J_3^{(n)}, J_\pm^{(n)}\} = (-\mathrm{i})(\pm J_\pm^{(n)}) \tag{4.6a}$$

$$\begin{aligned}\{J_+^{(n)}, J_-^{(n)}\} = -\mathrm{i}\bigg([H/2 + J_3]_{\gamma_1\gamma_2\cdots\gamma_n} \frac{\mathrm{d}[H/2 - J_3]_{\gamma_1\gamma_2\cdots\gamma_n}}{\mathrm{d}(H/2 - J_3)} \\ - [H/2 - J_3]_{\gamma_1\gamma_2\cdots\gamma_n} \frac{\mathrm{d}[H/2 + J_3]_{\gamma_1\gamma_2\cdots\gamma_n}}{\mathrm{d}(H/2 + J_3)}\bigg)\end{aligned} \tag{4.6b}$$

where the n-fold square bracket was applied

$$[x]_{\gamma_1\gamma_2\cdots\gamma_n} = \Big[[x]_{\gamma_1\gamma_2\cdots\gamma_{(n-1)}}\Big]_{\gamma_n} = \bigg[\Big[\cdots\Big[[x]_{\gamma_1}\Big]_{\gamma_2}\cdots\Big]_{\gamma_{(n-1)}}\bigg]_{\gamma_n}.$$

† For simplicity, we take the phases $\alpha_i(z_i, \bar{z}_i) = 0$, $i = 1, 2$.

This n-fold deformed algebra is denoted ${}^{(\mathrm{I})}\mathrm{SU}_{q_1q_2\cdots q_n,\hbar\to 0}(2)$, in order to be distinguished from that realized by the type-II q-oscillators, which is discussed in the following and is referred to as type II, denoted ${}^{(\mathrm{II})}\mathrm{SU}_{q_1q_2\cdots q_n,\hbar\to 0}(2)$.

The complex variables for the single-deformed system are provided in (2.9) and the n-fold deformed variables are

$$z_i^{(n)} = \frac{[N_i^{(n-1)}]_{\gamma_n}}{N_i} z_i^{(n-1)} \qquad \bar{z}_i^{(n)} = \frac{[N_i^{(n-1)}]_{\gamma_n}}{N_i} \bar{z}_i^{(n-1)}. \tag{4.7}$$

One can have the following algebraic relations for the type-II q-deformed system, when supplemented by the Hamiltonian H for the undeformed system,

$$\{H, J_\pm^{(n)}\} = 0 \qquad \{H, J_3^{(n)}\} = 0 \qquad \{J_3^{(n)}, J_\pm^{(n)}\} = (-\mathrm{i})(\pm J_\pm^{(n)}) \tag{4.8a}$$

$$\begin{aligned}\{J_+^{(n)}, J_-^{(n)}\} = -\mathrm{i}\bigg(&\langle H/2 + J_3\rangle_{\gamma_1\gamma_2\cdots\gamma_n} \frac{\mathrm{d}\langle H/2 - J_3\rangle_{\gamma_1\gamma_2\cdots\gamma_n}}{\mathrm{d}(H/2 - J_3)} \\ &- \langle H/2 - J_3\rangle_{\gamma_1\gamma_2\cdots\gamma_n} \frac{\mathrm{d}\langle H/2 + J_3\rangle_{\gamma_1\gamma_2\cdots\gamma_n}}{\mathrm{d}(H/2 + J_3)}\bigg)\end{aligned} \tag{4.8b}$$

where the angular bracket was applied, the single angular bracket being $\langle x\rangle_{\gamma_1} = [x]_{\gamma_1}$, the two-fold angular bracket $\langle x\rangle_{\gamma_1\gamma_2} = \left[[x]_{\gamma_1}^2/x\right]_{\gamma_2}/([x]_{\gamma_1}/x)$ and the n-fold angular bracket

$$\langle x\rangle_{\gamma_1\gamma_2\cdots\gamma_n} = \left[\frac{\langle x\rangle^2_{\gamma_1\gamma_2\cdots\gamma_{(n-1)}}}{x}\right]_{\gamma_n} \left(\frac{\langle x\rangle_{\gamma_1\gamma_2\cdots\gamma_{(n-1)}}}{x}\right)^{-1}$$

We denote the multi-deformed algebra by ${}^{(\mathrm{II})}\mathrm{SU}_{q_1q_2\cdots q_n,\hbar\to 0}(2)$, as already noted above. It is clear that that type *I* and *II* algebras coincide for the case $n = 1$, but remain different otherwise.

When one makes the canonical quantization in these multi-deformed systems of oscillators, follows the same procedure applied in last section to single-deformed system, one may get the quantum multi-deformed algebraic structures as before, however, we need to fix an ordering, so that we have H, the Hamiltonian of the undeformed system, and the generators $J_\pm^{(n)}$, $J_3^{(n)}$ as follows,

$$J_+^{(n)} = a_1^{(n)\dagger} a_2^{(n)} \qquad J_-^{(n)} = a_2^{(n)\dagger} a_1^{(n)} \qquad J_3^{(n)} = \tfrac{1}{2}(a_1^\dagger a_1 - a_2^\dagger a_2) \tag{4.9}$$

where the creation and annihilation operators are

$$a_i^{\dagger(n)} = \sqrt{\frac{\sinh(\gamma_n N_i^{(n-1)})}{\sqrt{\gamma_n \sinh\gamma_n} N_i^{(n-1)}}}\, a_i^{\dagger(n-1)} \qquad a_i^{(n)} = a_i^{(n-1)} \sqrt{\frac{\sinh(\gamma_n N_i^{(n-1)})}{\sqrt{\gamma_n \sinh\gamma_n} N_i^{(n-1)}}} \tag{4.10}$$

and $\gamma_n = \log q_n$, and $N_i^{(n-1)}$ is the particle number operator for the ith $(n-1)$-fold deformed q-oscillator. The algebra satisfied by these generators is

$$[H, J_\pm^{(n)}] = 0 \qquad [H, J_3^{(n)}] = 0 \qquad [J_3^{(n)}, J_\pm^{(n)}] = (-\mathrm{i})(\pm J_\pm^{(n)}) \tag{4.11a}$$

$$\begin{aligned}[J_+^{(n)}, J_-^{(n)}] = &[H/2 + J_3]_{\gamma_1\gamma_2\cdots\gamma_n} [H/2 - J_3 + 1]_{\gamma_1\gamma_2\cdots\gamma_n} \\ &- [H/2 - J_3]_{\gamma_1\gamma_2\cdots\gamma_n} [H/2 + J_3 + 1]_{\gamma_1\gamma_2\cdots\gamma_n}\end{aligned} \tag{4.11b}$$

This algebra is referred to be ${}^{(\mathrm{I})}\mathrm{SU}_{q_1q_2\cdots q_n,\hbar}(2)$, as the quantum counterparts of the multi-deformed algebra ${}^{(\mathrm{I})}\mathrm{SU}_{q_1q_2\cdots q_n,\hbar\to 0}(2)$.

The n-fold deformed algebra realized by type-II q-deformed oscillators at quantum level is spanned by the Hamiltonian H of the undeformed system and the generators $J_{\pm}^{(n)}$, $J_3^{(n)}$ as follows:

$$J_+^{(n)} = a_1^\dagger a_2^{(n)} \qquad J_-^{(n)} = a_2^\dagger a_1^{(n)} \qquad J_3^{(n)} = \tfrac{1}{2}(a_1^\dagger a_1 - a_2^\dagger a_2). \tag{4.12}$$

Here the creation and annihilation operators are

$$a_i^{\dagger(n)} = \frac{\sinh(\gamma_n N_i^{(n-1)})}{\sqrt{\gamma_n \sinh\gamma_n} N_i^{(n-1)}} a_i^{\dagger(n-1)} \qquad a_i^{(n)} = a_i^{(n-1)} \frac{\sinh(\gamma_n N_i^{(n-1)})}{\sqrt{\gamma_n \sinh\gamma_n} N_i^{(n-1)}}. \tag{4.13}$$

The algebra satisfied by the generators in (4.12) is referred to as ${}^{(\mathrm{II})}\mathrm{SU}_{q_1q_2\cdots q_n,\hbar}(2)$

$$[H, J_\pm^{(n)}] = 0 \qquad [H, J_3^{(n)}] = 0 \qquad [J_3^{(n)}, J_\pm^{(n)}] = \pm J_\pm^{(n)} \tag{4.14a}$$

$$\begin{aligned}[J_+^{(n)}, J_-^{(n)}] = {} & \langle H/2 + J_3\rangle_{\gamma_1\gamma_2\cdots\gamma_n} \langle H/2 - J_3 + 1\rangle_{\gamma_1\gamma_2\cdots\gamma_n} \\ & - \langle H/2 - J_3\rangle_{\gamma_1\gamma_2\cdots\gamma_n} \langle H/2 + J_3 + 1\rangle_{\gamma_1\gamma_2\cdots\gamma_n}.\end{aligned} \tag{4.14b}$$

When $q_n = 1$, the n-fold deformed algebras go back to the $(n-1)$-fold deformed algebras. When every parameter q_i is 1, these algebras reduce to SU(2). Hence, the n-fold deformed algebras are generalizations of the $\mathrm{SU}_q(2)$ algebra.

When the parameters q_i are roots of unity, one may have the chain truncated, i.e. a certain step of deformation may no longer be deformable [12]. For example, when q_i is the fourth ranked root of unity, then the chain stops here, because this q-oscillator is not deformable, i.e. the further deformed form of this oscillator is identical to the oscillator without this further deformation[14].

5. Discussions and remarks

In (I) and this paper we have realized $\mathrm{SU}_{q,\hbar\to 0}(2)$ algebra in a classical system with the q-deformed oscillators of two different types by means of Poisson brackets, and then through canonical quantization, we have obtained its quantum counterpart, $\mathrm{SU}_{q,\hbar}(2)$ by means of Lie brackets. A set of j-representation of the quantum algebra $\mathrm{SU}_q(2)$ is constructed based on the type-II q-deformed oscillators, and it is pointed out that with a given metric, $|jm\rangle$ are a set of orthogonal bases. We have also investigated multi-deformations of the q-oscillators and the multi-deformed algebras expressed in Poisson and Lie brackets for both the type-I and type-II q-oscillators respectively. The structures of the Hopf algebra and the quantum Yang–Baxter equation for $\mathrm{SU}_{q,\hbar}(2)$ have also been realized in the quantum q-oscillator systems. Although we have not touched the Hopf algebraic structure of the $\mathrm{SU}_{q,\hbar\to 0}(2)$, it is reasonable to expect that there should be some non-trivial Hopf algebraic structure relevant to $\mathrm{SU}_{q,\hbar\to 0}(2)$ at the classical level. We would leave this subject for further investigations.

It is of interest to see that the q-deformation of type II discussed in section 2 is also a kind of Beltrami transformation or quasiconformal deformation [13], as was the case in (I) for the q-deformation of type I. Actually, from (2.33), one has Beltrami coefficients in the phase space V of

$$\mu_i(z_i, \bar{z}_i) = \bar{\partial} z_i' / \partial z_i' \tag{5.1}$$

satisfying

$$|\mu(z_i, \bar{z}_i)| < 1. \tag{5.2}$$

Since the Beltrami deformation has an inverse, the transformation between the old Beltrami deformation given in (I) and the new deformation described above can easily be combined to form a single Beltrami transformation. Hence, the transformation between the q-deformed oscillators of type I and type II is also a Beltrami transformation.

For the case of multi-deformations, as quasi-conformal transformations in general form a group in certain sense, there might be an additional algebraic structure on the chain of the multi-deformations. It is also reasonable to expect that the multi-deformations should be relevant to certain algebraic structures of the Hopf type and of the Yang–Baxter type.

When the parameter q is a root of unity, interesting mathematical and physical results for type I have been obtained; these are partially dealt with in [14]. We will also explain this issue for the both cases in detail in the forthcoming paper [12].

In (I) and this paper, we have dealt with the deformations of the SU(2) algebra only. However, it is clear that not only the method but also the main results have a general meaning. First, just as one can easily transfer SU(2) to SU(1, 1) by means of the Weyl unitary trick, the classical and quantum q-deformations of the SU(1, 1) algebra may be reached from that of the SU(2) algebra. For more general cases, such as A_N, B_N, C_N and D_N algebras as well as their non-compact counterparts, similar realizations exist for the q-deformed algebras at both classical and quantum levels, which will be analysed in detail in a separate paper [15].

Finally, it should be remarked that the approach proposed in [6] and (I) as well as in this paper to realize the q-deformed algebras is based on a set of deformed and undeformed oscillators on undeformed symplectic space. However, there is an alternative approach, proposed by Shao-Ming Fei and Han-Ying Guo [16], to realize the q-deformed algebras at both classical and quantum levels by deforming the symplectic geometry rather than the observables. It is of course meaningful to see whether their approach can be applied to the harmonic oscillators. This and other relevant subjects are under consideration.

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References

[1] Drinfeld V G 1986 *Proc. Int. Conf. Math., Berkeley, CA* ed A M Gleason (Providence, RI: American Mathematical Society)

[2] Jimbo M 1985 *Lett. Math. Phys.* **10** 63; 1986 *Commun. Math. Phys.* **102** 537

[3] Manin Yu I 1988 Quantum groups and non-commutative geometry *Preprint* University of Montreal CRM-1561 (1988)

[4] Faddeev L D, Reshetikhin N and Takhtajan L A 1987 Quantization of Lie groups and Lie algebras *Preprint* LOMI Leningrad

[5] Yang C N 1967 *Phys. Rev. Lett.* **19** 1312
Baxter R J 1982 *Exactly Solved Models in Statistical Mechanics* (New York: Academic)

[6] Wei Chen, Zhe Chang and Han-Ying Guo 1990 Classical system of q-deformed harmonic oscillators and their $\hbar$-quantization *Acta Phys. Sinica* to appear (in Chinese)

[7] Zhe Chang, Wei Chen and Han-Ying Guo 1990 *J. Phys. A: Math. Gen.* **23** 4185–4190

[8] Biedenharn L C 1989 *J. Phys. A: Math. Gen.* **22** L873

[9] Macfarlane A J 1989 *J. Phys. A: Math. Gen.* **22** 4581

[10] Ng Y J 1990 *J. Phys. A: Math. Gen.* **23** 1023

[11] Xing-Chang Song 1990 *Preprint* CCAST CCAST-90-7

[12] Zhe Chang, Wei Chen, Han-Ying Guo and Hong Yan 1990 $SU_{q,\hbar\to 0}(2)$ and $SU_{q,\hbar}(2)$, the classical and quantum q-deformations of SU(2) algebra (III): q is the root of unity *Preprint* Institute of Theoretical Physics, Academia Sinica, and CCAST ASITP-90-22, CCAST-90-18

[13] See for example Krushkal S L 1979 *Quasiconformal Mappings and Riemann Surfaces* (New York: Winston)

[14] Hong Yan, Wei Chen, Zhe Chang and Han-Ying Guo 1990 A kind of q-deformed oscillators and their non-bosonic excitations when $q^k = 1$ *Acta Phys. Sinica* to appear (in Chinese)

[15] In preparation

[16] Shao-Ming Fei and Han-Ying Guo 1990 Symplectic geometry of $SU_{q,\hbar\to 0}(2)$ and $SU_{q,\hbar\to 0}(2)$ algebras *Preprint* Institute of Theoretical Physics, Academia Sinica, and CCAST ASITP-90-19, CCAST-90-16 (to appear in J. Phys. A: Math. Gen.)

[17] Alvarez-Gaumé L, Sierra G and Gomez C 1989 Topics in Conformal Field Theory *CERN Report* CERN-TH.5540/89